

A dichotomy for the kernel by H -walks problem in digraphs

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Abstract

Let $H = (V_H, A_H)$ be a digraph which may contain loops, and let $D = (V_D, A_D)$ be a loopless digraph with a coloring of its arcs $c : A_D \rightarrow V_H$. An H -walk of D is a walk (v_0, \dots, v_n) of D such that $(c(v_{i-1}, v_i), c(v_i, v_{i+1}))$ is an arc of H , for every $1 \leq i \leq n - 1$. For $u, v \in V_D$, we say that u reaches v by H -walks if there exists an H -walk from u to v in D . A subset $S \subseteq V_D$ is a kernel by H -walks of D if every vertex in $V_D \setminus S$ reaches by H -walks some vertex in S , and no vertex in S can reach another vertex in S by H -walks.

A panchromatic pattern is a digraph H such that every arc-colored digraph D has a kernel by H -walks. In this work, we prove that every digraph H is either a panchromatic pattern, or the problem of determining whether an arc-colored digraph D has a kernel by H -walks is NP -complete.

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1 Introduction

Sands, Sauer and Woodrow proved in [17] a beautiful result stating that every digraph whose arcs are colored with two colors has a kernel by monochromatic paths. Since then, the existence of kernels by monochromatic paths has been studied both in general digraphs [8, 9, 10] and in tournaments [7, 12, 18].

Linek and Sands generalized in [16] the notion of kernel by monochromatic paths in the following way. If H is a digraph, possibly with loops, we will say that the digraph D is H -arc-colored if the set of arcs of D has been colored with the vertices of H ; we will usually denote this coloring with c . An H -walk is a walk (v_0, \dots, v_n) in D such that $(c(v_i, v_{i+1}), c(v_{i+1}, v_{i+2}))$ is an arc of H ; we say that v_0 reaches v_n by H -walks. A kernel by H -walks of D is a subset S of V_D such that it is *independent by H -walks* (there are no H -walks between vertices in S) and *absorbent by H -walks* (for every vertex u in $V_D \setminus S$ there is a vertex v in S such that u reaches v by H -walks).

If $V_H = \{x, y\}$ and $A_H = \{(x, x), (y, y)\}$, then a kernel by H -walks in a digraph D is simply a *kernel by monochromatic paths* (observe that every monochromatic walk contains a monochromatic path), and hence, the aforementioned theorem of Sands, Sauer and Woodrow states that every H -arc-colored digraph has a kernel by H -walks (for this particular choice of H). In this context, a very natural question arises. Which are the digraphs H such that every H -arc-colored digraph has a kernel by H -walks? Arpin and Linek stated this question in [1], and found some digraphs having this property, as well as some digraphs not having this property. A *panchromatic pattern* is a digraph H such that every H -arc-colored digraph has a kernel by H -walks. Based on the work of Arpin and Linek, Galeana-Sánchez and Strausz characterized all the panchromatic patterns in [11]. Again, a natural question arose. How hard is to determine the existence of a kernel by H -walks if H is not a panchromatic pattern? For a digraph H , define the *kernel by H -walks problem* to be the decision problem of determining whether a digraph D has a kernel by H -walks. Considering that H can be any digraph, and the panchromatic patterns are a very restricted class, it is natural to think that there are some digraphs H having a polynomial time solvable kernel by H -walks problem, as well as some digraphs H having an NP -complete kernel by H -walks problem. It really comes as a surprise that the former case never happens. The following theorem is the main result of this work.

Theorem 1. *If H is a digraph, then either H is a panchromatic pattern (and hence the kernel by H -walks problem is constant time solvable), or the kernel by H -walks problem is NP -complete.*

There are only a handful of articles dealing with the complexity of the many variations of the kernel problem. Chvátal proved in 1973 [5] that the kernel problem is NP -complete; in 1981 [6], Fraenkel proved that the kernel problem remained NP -complete even when restricted to planar graphs with $\Delta^+, \Delta^- \leq 2$ and $\Delta \leq 3$; in 2014 [14], Hell and Hernández-Cruz proved that the 3-kernel problem is NP -complete, even when restricted to digraphs homomorphic to a (directed) 3-cycle with circumference 6, and, as a consequence, the kernel problem remains NP -complete even when restricted to 3-colorable digraphs. On the other hand, Bang-Jensen, Guo, Gutin and Volkman proved in 1997 [3] that the kernel problem is polynomial time solvable for locally semicomplete digraphs; in [14] it is proved that the 3-kernel problem is polynomial time solvable for semicomplete multipartite digraphs.

This is the first work dealing with the complexity of a generalization of the kernel problem different from the 3-kernel problem. In fact, although the kernel by monochromatic paths has been widely studied, its complexity remained unknown until now.

We refer the reader to [2] and [4] for general concepts. Let $D = (V_D, A_D)$ be a digraph with set of vertices V_D and set of arcs A_D . A *loop* is an arc of the form (v, v) ; a vertex v is *looped* if (v, v) is a loop of D , and it is *loopless* otherwise. A digraph is *looped* (*loopless*) if all its vertices are looped (loopless).

A *digon* is an arc (x, y) of D such that (y, x) is also an arc of D ; given the symmetric nature of the definition, we will sometimes refer to an unordered pair of vertices $\{x, y\}$ as a digon. An arc (x, y) of D is *asymmetric* if $(y, x) \notin A_D$. We will say that a subset S of V_D is a strong clique if every pair of vertices in S is a digon.

Given a family \mathcal{F} of digraphs, we say that a digraph D is \mathcal{F} -free if no member of \mathcal{F} appears as an (homomorphic copy of an) induced subdigraph of D . A property \mathcal{P} of a digraph D is *hereditary* if every induced subdigraph of D also has the property \mathcal{P} . For example, the property of being \mathcal{F} -free is a hereditary property.

All walks, paths and cycles are considered to be directed unless otherwise stated. The circumference of a digraph is the length of its longest cycle, and it is defined to be zero for acyclic digraphs. For a positive integer k , a k -cycle

is a cycle of length k .

The remainder of this work is organized as follows. In Section 2 we show that the kernel by H -walks problem is in the class NP , by presenting a polynomial time algorithm to verify reachability by H -walks. Section 3 is devoted to obtain a characterization of panchromatic patterns in terms of a finite family of forbidden induced subdigraphs. In order to prove NP -hardness of the kernel by H -walks problem for every H which is not a panchromatic pattern, we consider three cases for H ; the polynomial reduction for the most complex case is constructed in Section 4. In Section 5 we deal with the two remaining cases to complete the proof of Theorem 1. Section 6 is devoted to present our concluding remarks and future lines of work.

2 A reachability algorithm

In order to prove that the kernel by H -walks problem is in NP , it suffices to show that, given an instance (D, H) of the problem, and a subset S of V_D , it can be verified in polynomial time whether S is a kernel by H -walks of D . It is clear that this can be achieved through an algorithm that finds all the vertices reached by H -walks from a given vertex v , in polynomial time. We only have to run this algorithm from every vertex of D .

In this section we will propose such algorithm and prove that it runs in polynomial time. The proposed algorithm is based on BFS. The main difference is that, instead of vertices, our queue will have pairs (x, c) , where x is a vertex and c is the color of the arc which was explored in order to reach x . Each of this pairs may join the queue at most once, but the same vertex may be considered several times. This will allow us to know through which color a vertex was reached, and thus, to find H -walks that use some vertex several times. Once a pair has joined the queue, it will be painted black, so it will not be explored again in the future, just as in BFS. But, unlike BFS, when we are considering an out-neighbor y of a vertex x , we will check whether y is uncolored, and also if the color $c(x, y)$ of the arc (x, y) is compatible in H with the color c we used to reach x (when (x, c) is the current head of the queue); in other words, we need that $(c, c(x, y)) \in A_H$.

Now, we present the pseudo-code for our algorithm. The only two structures that we will use are a queue Q and a set R . The input will be an H -arc-colored digraph $D(v)$ with a distinguished vertex v , the “root” of our search. We would like to point out that we are not trying to optimize the

running time of the algorithm, and probably there are better implementations of the same idea we are using, but we tried to keep the algorithm as clear and simple as possible.

Algorithm 1: Reachability by H -walks algorithm

Input: An H -arc-colored digraph $D(v)$.

Output: The set R of all vertices reached by H -walks from v

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1  $Q \leftarrow \emptyset, R \leftarrow \emptyset$ 
2  $R \leftarrow R \cup \{v\}$ 
3 for  $c \in V_H$  do
4   | color  $(v, c)$  black
5   | append  $(v, c)$  to  $Q$ 
6 end
7 while  $Q$  is nonempty do
8   | consider the head  $(x, c)$  of  $Q$ 
9   | if  $x$  has an out-neighbor  $y$  such that  $(y, c(x, y))$  is uncolored and
      |  $(c, c(x, y)) \in A(H)$  then
10  |   | color  $(y, c(x, y))$  black
11  |   |  $R \leftarrow R \cup \{y\}$ 
12  |   | append  $(y, c(x, y))$  to  $Q$ 
13  | else
14  |   | remove  $(x, c)$  from  $Q$ 
15  | end
16 end
17 return  $R$ 

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Theorem 2. *Let $D(v)$ be an H -arc-colored digraph. A vertex $u \in V_D$ is reached from v by H -walks in D if and only if $u \in R$, where R is the output of Algorithm 1*

Proof. We will use an inductive argument to prove that every vertex in R is reached from v by H -walks in D . Let u be a vertex in R . It follows from steps 8 and 9 of Algorithm 1 that $(u, c(x, u))$ joined Q while exploring the pair (x, c) , where x is an in-neighbor of u already in R , and such that $(c, c(x, u)) \in A(H)$.

Assume that, for every pair (y, c) that joined Q before $(u, c(x, u))$, either $y = v$, or y is reached from v by an H -walk, W , such that the color of the last

arc of W is c . Since (x, c) joined Q before $(u, c(x, u))$, there exists an H -walk, W_x , from v to x with the aforementioned property. Clearly $W_x \cup (x, u)$ is an H -walk in D and the color of its last arc is $c(x, u)$. The desired result now follows from the Second Principle of Mathematical Induction.

It remains to show that every vertex reached by H -walks from v in D belongs to R . Clearly, $v \in R$. Now, let u be a vertex in D such that $u \neq v$, and an H -walk, W , from v to u exists in D . Let $W = (v = x_0, \dots, x_n = u)$. We will prove by induction on the length of W , $\ell(W)$, that $(u, c(x_{n-1}, u))$ is colored black during the running of Algorithm 1. If $\ell(W) = 1$, then u is an out-neighbor of v and the result is immediate. Suppose that $\ell(W) = n$. By the induction hypothesis, $(x_{n-1}, c(x_{n-2}, x_{n-1}))$ is colored black while Algorithm 1 is running. Thus, $(x_{n-1}, c(x_{n-2}, x_{n-1}))$ is the head of Q at some point, and the pair $(u, c(x_{n-1}, u))$ is considered in step 9. Since W is an H -walk, the arc $(c(x_{n-2}, x_{n-1}), c(x_{n-1}, u))$ is in H , and hence, $(u, c(x_{n-1}, u))$ is explored (and thus colored black) in this step, unless it has been previously colored black. In either case, the desired pair is colored black, and thus, it follows from step 11 in Algorithm 1 that $u \in R$. \square

Let us make a brief running time analysis for Algorithm 1. Steps 1 and 2 are executed a constant number of times. Steps 4 and 5 are performed once for each vertex in H , this is $o(V_H)$. For every arc (x, y) , the pair $(y, c(x, y))$ may join Q at most once and, while in Q , every arc with tail y should be considered. Hence, the number of times steps 9–15 are performed is a linear function of $\sum_{v \in V_D} d_D^-(v) \cdot d_D^+(v) \leq \sum_{v \in V_D} \Delta_D^- \cdot d_D^+(v) = \Delta_D^- \cdot |A_D| \leq |V_D| \cdot |A_D|$. Hence, the running time of Algorithm 1 is $o(|V_H| + |V_D| \cdot |A_D|)$, which is polynomial.

Moreover, Algorithm 1 can be modified to omit the $|V_H|$ from the running time. Just add directly $(x, c(v, x))$ to Q for every out-neighbor of v instead of performing steps 3–6; this is done at most $|V_D|$ times. Thus, the running time of this modified version of Algorithm 1 is $o(|V_D| \cdot |A_D|)$.

As we have discussed at the beginning of this section, the following result is already proved.

Corollary 3. *The problem of determining whether an H -arc-colored digraph has a kernel by H -walks is in NP.*

3 Minimal obstructions for panchromaticity

Given an m by m matrix M over $\{0, 1, *\}$, an M -partition of a digraph D is a partition of its vertex set into parts V_1, \dots, V_m such that each vertex in V_i must (respectively must not) dominate each vertex in V_j if $M_{i,j} = 1$ (respectively $M_{i,j} = 0$); there are no restrictions if $M_{i,j} = *$. When $i = j$, V_i is a strong clique (respectively an independent set) if $M_{i,i} = 1$ (respectively $M_{i,i} = 0$). If M is a symmetric matrix, a similar definition can be given for undirected graphs. An excellent survey on the subject of matrix partitions of graphs and digraphs is due to Hell [13].

The M -partition problem asks whether a given digraph D admits an M -partition. Particular choices of matrices M yield very well known problems in the undirected graph case, e.g., a $\begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}$ -partition is simply a 2-coloring and a $\begin{pmatrix} 0 & * \\ * & 1 \end{pmatrix}$ -partitionable graph is a split graph.

It is easy to observe that having an M -partition is a hereditary property, and hence, it can be characterized through a set of minimal forbidden induced subdigraphs. For a matrix M , we define an M -obstruction to be any digraph not having an M -partition. An M -obstruction, D , is minimal if any induced proper subdigraph of D has an M -partition. Of course, it is direct to verify that a digraph admits an M -partition if and only if it does not contain any minimal M -obstruction as an induced subdigraph.

Let M_1 and M_2 be the 2×2 matrices

$$M_1 = \begin{pmatrix} 1 & 1 \\ * & 1 \end{pmatrix} \quad M_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Based on the work of Arpin and Linek [1], Galeana-Sánchez and Strausz proved that a digraph is a panchromatic pattern if and only if it is a looped M_1 -partitionable or M_2 -partitionable digraph, [11]. Again, it is easy to observe that having an M_1 -partition or an M_2 -partition is a hereditary property. We define a *panchromatic obstruction* to be a digraph having neither an M_1 -partition nor an M_2 -partition. A panchromatic obstruction is *minimal* if every induced subdigraph admits an M_1 -partition or an M_2 -partition. The aim of this section is to characterize all the minimal panchromatic obstructions. To this end, we will use the characterization given by Hell and Hernández-Cruz in [15] of all the M_1 -minimal obstructions and M_2 -minimal obstructions.

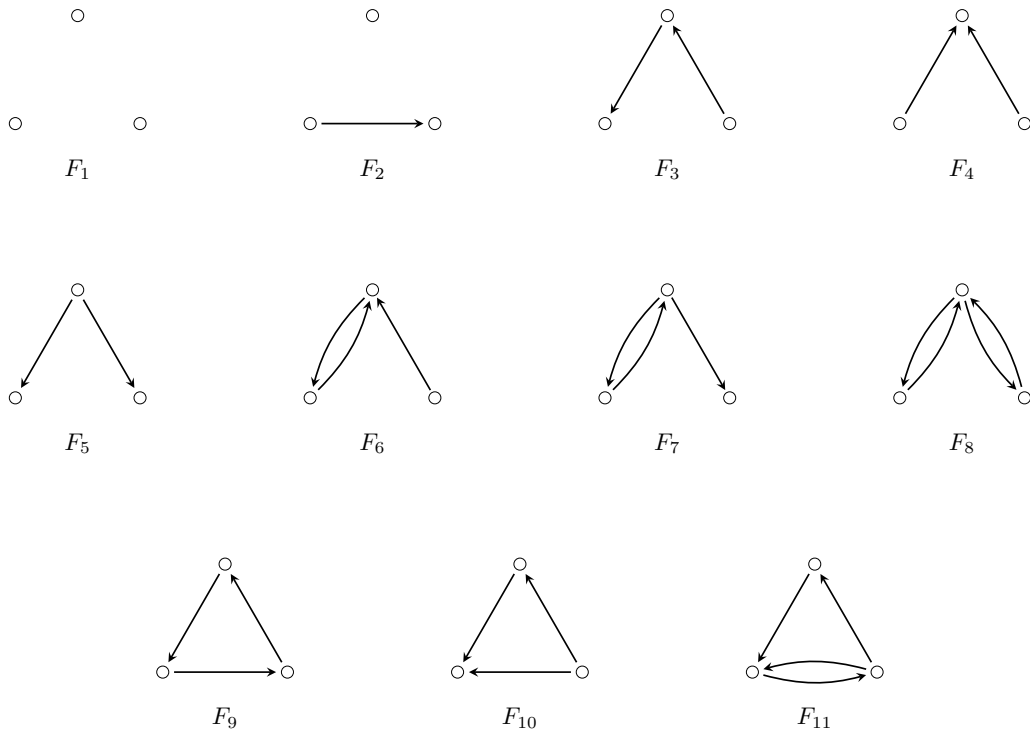


Figure 1: Minimal panchromatic obstructions.

Theorem 4. *Let M_1 and M_2 be the 2×2 matrices defined above. Then,*

1. *The minimal M_1 -obstructions are, an independent set of two vertices, and the digraphs F_9, F_{10} and F_{11} in Figure 1.*
2. *The minimal M_2 -obstructions are, an asymmetric arc, and the digraphs F_1 and F_8 in Figure 1.*

Let \mathcal{F} be the set $\{F_i\}_{i=1}^{11}$, depicted in Figure 1. Observe that the digraphs in \mathcal{F} are minimal panchromatic obstructions. Indeed, the digraphs F_9, F_{10} and F_{11} are minimal M_2 -obstructions containing at least one asymmetric arc, which makes them also M_2 -obstructions. Similarly, F_1 is a minimal M_2 -obstruction containing an independent set of size 2, which makes it an M_1 -obstruction. Each of the remaining digraphs in \mathcal{F} contains an asymmetric arc and an independent set of cardinality 2, and hence is both an M_1 -obstruction and an M_2 -obstruction. These obstructions are minimal since any digraph on 2 vertices is either M_1 -partitionable or M_2 -partitionable. Hence, every digraph in \mathcal{F} is a minimal panchromatic obstruction. Our next result shows that these are the only minimal panchromatic obstructions.

Theorem 5. *If D is a digraph, then D admits an M_1 -partition or an M_2 -partition if and only if it is \mathcal{F} -free.*

Proof. For the necessity, we have already observed in the previous paragraph that the digraphs in \mathcal{F} are minimal panchromatic obstructions. Therefore, a digraph D admitting an M_1 -obstruction or an M_2 -obstruction cannot contain any of the digraphs in \mathcal{F} .

For the sufficiency, we will proceed by contrapositive. Let D be a panchromatic obstruction.

The minimal M_1 -obstructions and the minimal M_2 -obstructions on 3 vertices are in the set \mathcal{F} . Thus, if D contains any of them, we are done. Otherwise, in order for D to be a panchromatic obstruction, D should contain the only minimal M_1 -obstruction on two vertices, (an independent set of size two), and the only minimal M_2 -obstruction on two vertices, (an asymmetric arc).

Let $\{v_1, v_2\}$ be an independent set, and let (v_3, v_4) be an asymmetric arc of D . First suppose that $\{v_1, v_2\} \cap \{v_3, v_4\} \neq \emptyset$. This intersection contains precisely one vertex. Hence, the set $\{v_1, v_2\} \cup \{v_3, v_4\}$ has cardinality 3, and it is easy to verify that it induces the digraph F_i for some $i \in \{2, 3, 4, 5, 6, 7\}$. Thus, we are done.

Else, $\{v_1, v_2\} \cap \{v_3, v_4\} = \emptyset$. If there are no arcs, or there is an asymmetric arc, between $\{v_1, v_2\}$ and $\{v_3, v_4\}$, then we can choose an asymmetric arc with an end in $\{v_1, v_2\}$, or we can choose an independent set of size two sharing a vertex with (v_3, v_4) , as in the previous case. Hence, every arc between $\{v_1, v_2\}$ and $\{v_3, v_4\}$ is present, and is a digon. In this case, the set $\{v_1, v_2, v_3\}$ induces the digraph F_8 .

Since the cases are exhaustive and in all of them we obtain the existence of an induced subdigraph of D isomorphic to a member of \mathcal{F} , the proof is complete. \square

Corollary 6. *Panchromatic patterns can be recognized in polynomial time.*

Proof. To verify if a digraph D is a panchromatic pattern, it suffices to check whether D has induced subdigraphs isomorphic to some member of \mathcal{F} . Using a brute force approach, this amounts to verify every subset of 3 vertices of V_D , which can be done in roughly $o(|V_D|^3)$ -time. \square

4 The reduction for F_1, F_5, F_7 and F_8

Before proving our main result, we will construct a polynomial reduction scheme that will be used to prove that any looped digraph H containing one of F_1, F_5, F_7 or F_8 as an induced subdigraph, has an NP -complete kernel by H -walks problem. We will reduce it from the k -colouring problem for graphs (k -COL). We would like to point out that we will not construct a single polynomial reduction, but a family of reductions that, depending on a part of the vertex gadget, will work for different choices of H .

For the remainder of this section H will denote a looped digraph containing F_1, F_5, F_7 or F_8 as an induced subdigraph. For a given graph G , we will construct a digraph D_G such that, the graph G is k -colorable if and only if D_G has a kernel by H -walks. In the following construction, *red*, *green* and *blue* will be the vertices of the induced copy of F_i , $i \in \{1, 5, 7, 8\}$, in H ; the set $\{green, blue\}$ will be an independent set of H .

Let G be a graph, and consider any of its acyclic orientations, \vec{G} . Let $k \geq 3$ be a fixed integer. For each vertex v of G , construct a k -cycle $C_v = (x_{v1}, \dots, x_{vk}, x_{v1})$, and color each of its arcs green. Also, create a copy F_v of an H -arc-colored digraph F not having a kernel by H -walks, add all the arcs from F_v to C_v , and color them green. The resulting colored digraph is the

gadget for the vertex v . Observe that this gadget depends on the choice of F .

Now, for every arc (u, v) of \vec{G} , and for every $1 \leq i \leq k$, create a directed 4-cycle $Q_{(u,v)i}$ with arcs $(x_{(u,v)i}, y_{(u,v)i})$ and $(z_{(u,v)i}, w_{(u,v)i})$, colored green, and $(y_{(u,v)i}, z_{(u,v)i})$ and $(w_{(u,v)i}, x_{(u,v)i})$ colored blue. Finally, add the arcs $(x_{ui}, x_{(u,v)i})$ and $(w_{(u,v)i}, x_{vi})$ colored blue. Let D_G be the resulting colored digraph. Notice that the 4-cycle $Q_{(u,v)i}$ could be replaced by any even cycle colored with alternating colors, or with a blue colored digon. Nonetheless, we prefer to keep the 4-cycle because, if F does not have digons, then D_G neither has digons. Figure 2 shows the construction of the gadget for the arc (u, v) when $k = 3$. The dashed lines represent green arcs and the solid lines represent blue arcs.

Lemma 7. *Let G be a graph. If D_G has a kernel by H -walks, K , then,*

1. *For every vertex v of G , precisely one vertex of C_v belongs to K .*
2. *If (u, v) is an arc of \vec{G} , then,*
 - (a) *If $x_{ui} \in K$, then $x_{vi} \notin K$.*
 - (b) *If $x_{vi} \in K$, then $x_{ui} \notin K$.*

Proof. Let v be any vertex of G . Since F_v does not have a kernel by H -walks, then at least one vertex of F_v must be absorbed by a vertex in $K \setminus V(F_v)$. By construction, $\partial^+(F_v) = V(C_v)$, and all the arcs going out from C_v are blue, hence, $V(C_v) \cap K \neq \emptyset$. On the other hand, C_v is a monochromatic cycle, and thus $|K \cap V(C_v)| \leq 1$. Therefore, $|K \cap V(C_v)| = 1$.

Now, suppose that $x_{ui} \in K$. Since $(x_{ui}, x_{(u,v)i})$ is an arc of D , we have $x_{(u,v)i} \notin K$. But $Q_{(u,v)i}$ is a 4-cycle with alternating colors blue and green, hence, locally, K behaves as a regular kernel. Thus, $y_{(u,v)i}$ and $w_{(u,v)i}$ must belong to K in order to absorb $x_{(u,v)i}$ and $z_{(u,v)i}$ by H -walks, respectively. Since $(w_{(u,v)i}, x_{vi})$ is an arc of D_G , we have $x_{vi} \notin K$.

Similarly, if $x_{vi} \in K$, then $w_{(u,v)i}$ is absorbed by H -walks by x_{vi} , and hence, it cannot belong to K . This fact, together with the structure of $Q_{(u,v)i}$ implies that $z_{(u,v)i}$ and $x_{(u,v)i}$ belong to K . Since $(x_{ui}, x_{(u,v)i})$ is an arc of D_G , we have $x_{ui} \notin K$. \square

Lemma 8. *Let G be a graph, and let I be an independent by H -walks subset of $V(D_G)$ such that $I \cap V(C_v) \neq \emptyset$ for every $v \in V(G)$. Let (u, v) be an arc*

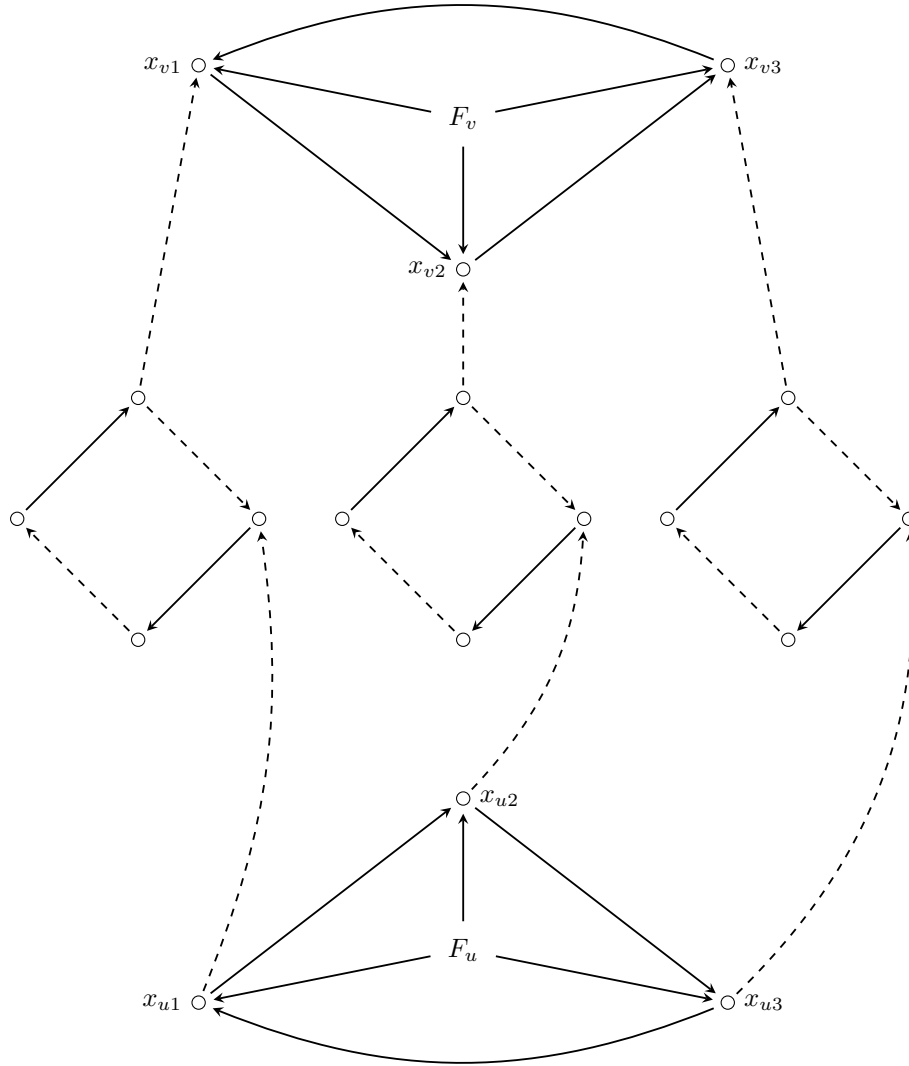


Figure 2: Gadget for the arc (u, v)

of \vec{G} , and let $I \cap V(C_u) = \{x_{ui}\}$ and $I \cap V(C_v) = \{x_{vj}\}$. If for every arc (u, v) of \vec{G} we have $i \neq j$, then I can be extended to be a kernel by H -walks of D_G .

Proof. Consider I as in the hypothesis. By construction, every vertex in the gadget for every $v \in V(G)$ is absorbed by the only vertex in $I \cap V(C_v)$.

Let (u, v) be an arc in \vec{G} and suppose that $I \cap V(C_u) = \{x_{ui}\}$. For every $j \neq i$, add $x_{(u,v)j}$ and $z_{(u,v)j}$ to I ; also, add $y_{(u,v)i}$ and $w_{(u,v)i}$ to I . Repeat this process with every arc of \vec{G} and let K be the resulting subset of $V(D_G)$. In the former case, $y_{(u,v)j}$ and $w_{(u,v)j}$ are absorbed by H -walks by $z_{(u,v)j}$ and $x_{(u,v)j}$, respectively. In the latter case, $x_{(u,v)i}$ and $z_{(u,v)i}$ are absorbed by H -walks by $y_{(u,v)i}$ and $w_{(u,v)i}$, respectively.

For the independence by H -walks of K , first observe that, except from the arcs from F_v to C_v , every arc coming from and going to C_v in D_G is blue. Since all the arcs from F_v to C_v , and all the arcs in C_v are green, the set $\{x_{ui}, x_{vj}\}$ is independent by H -walks, for every pair of different vertices $u, v \in V(G)$. Recalling the construction of K in the preceding paragraph, $x_{(u,v)i} \notin K$ when $x_{ui} \in K$. Also, this is the only case where $w_{(u,v)i} \in K$, but by the choice of I , we have $x_{vi} \notin K$. Hence K is independent by H -walks. \square

Lemma 9. *If G is a graph, then G is k -colorable if and only if D_G has a kernel by H -walks.*

Proof. Let $c : V(G) \rightarrow \{1, \dots, k\}$ be a k -coloring of G . Consider the subset $I \subset V(D_G)$ defined as $\{x_{vi} : v \in V(G), c(v) = i\}$. Clearly, I is a set fulfilling the conditions of Lemma 8, so it can be extended in a unique way to a kernel by H -walks, K , of D_G .

Conversely, let K be a kernel by H -walks of D_G . By Lemma 7, for every vertex $v \in V(G)$, there is precisely one vertex in $K \cap V(C_v)$, say, x_{vi} . If we define a k -coloring, c , of G as $c(v) = i$ for every $v \in V(G)$, then Lemma 7 implies that adjacent vertices receive different colors. Thus, c is a proper k -coloring of G . \square

5 Main Results

We have already proved in Section 2 that the kernel by H -walks problem is in NP . Hence, in order to prove that it is NP -complete for a particular choice of H , it suffices to prove its NP -hardness. As usual, we will achieve

this through a polynomial reduction. To prove the dichotomy we claim, we must show that every digraph H which is not a panchromatic pattern, has an NP -hard kernel by H -walks problem. Since every panchromatic pattern is a looped digraph, we will begin discussing the digraphs H which are missing at least one loop.

Let H be a digraph, and let red be a vertex of H without loops. Clearly, if we consider any digraph D , and color each of its arcs red to obtain the H -arc-colored digraph D' , then the only H -walks of D' are its arcs. Thus, a set $K \subseteq V_D$ is a kernel of D if and only if it is a kernel by H -walks of D' . This argument describes a linear reduction of the kernel by H -walks problem from the kernel problem. Moreover, as discussed on Section 1, D can be chosen to be 3-colorable, or planar with $\Delta^-, \Delta^+ \leq 2$ and $\Delta \leq 3$, and the problem remains NP -complete. So, we have proven the following result.

Proposition 10. *Let H be a digraph with at least one loopless vertex. The problem of determining whether an arc-colored digraph D has a kernel by H -walks is NP -complete, even when restricted to 3-colorable digraphs, or to planar digraphs with $\Delta^-, \Delta^+ \leq 2$ and $\Delta \leq 3$.*

Now, we can consider only looped digraphs H . Although the forbidden induced subdigraph characterization given in Section 3 deals with loopless digraphs, it is clear that the same characterization works with looped digraphs, we just have to consider the looped version of each digraph in the family \mathcal{F} . Our next result deals with the digraphs H containing a member of the family \mathcal{F} , other than F_1, F_5, F_7 or F_8 , as an induced subdigraph.

Theorem 11. *Let H be a looped digraph containing three vertices, red, blue and green, such that there is an asymmetric arc from red to green, and the arc from red to blue is missing. The problem of determining whether an H -arc-colored digraph D has a kernel by H -walks is NP -complete, even when restricted to planar bipartite digraphs with $\Delta^-, \Delta^+ \leq 2$, and $\Delta \leq 3$.*

Proof. We will reduce it from the kernel problem, which is known to be NP -complete even when restricted to planar digraphs with $\Delta^-, \Delta^+ \leq 2$, and $\Delta \leq 3$.

Let D be a planar digraph with the aforementioned degree restrictions. We will construct a new H -arc-colored digraph D' in the following way. Subdivide every arc (x, y) of D by adding the intermediate vertex $v_{(x,y)}$, and create an additional vertex $v'_{(x,y)}$ along with the arc $(v_{(x,y)}, v'_{(x,y)})$. For every

arc (x, y) of D , color red, green and blue the arcs $(x, v_{(x,y)})$, $(v_{(x,y)}, y)$ and $(v_{(x,y)}, v'_{(x,y)})$ of D' , respectively. Clearly, the digraph D' is planar, bipartite, and have the desired degree constraints. We will prove that D has a kernel if and only if D' has a kernel by H -walks.

Let S and T be the subsets of V_D' defined as

$$S = \{v'_{(x,y)} : (x, y) \in A_D\},$$

$$T = \{v_{(x,y)} : (x, y) \in A_D\}.$$

Since every vertex in S has out-degree equal to zero, S must be contained in every kernel by H -walks of D' . Moreover, every vertex in T is absorbed by some vertex in S , so T does not intersect any kernel by H -walks of D .

Claim 1. *For every vertex $x \in V_D \cap V_{D'}$, the vertices reached from x in D' by H -walks are precisely the vertices in the set*

$$R = N_D^+(x) \cup \{v_{(x,y)} : y \in N^+(x)\}.$$

Suppose first that D has a kernel K . We affirm that $K' = K \cup S$ is a kernel by H -walks for D' . We have already observed that every vertex of T is absorbed by H -walks by K' . Let x be a vertex in $V_{D'} \cap V_D$. If $x \notin K$, then there exists $y \in K$ such that $(x, y) \in A_D$. It follows from Claim 1 that x is absorbed by H -walks by y . Hence K' is absorbent by H -walks in D' . For the independence by H -walks of K' , notice that the vertices in S have zero out-degree, so they cannot reach any other vertex. Now, for the vertices in K , it follows from Claim 1 that they cannot reach the vertices in S by H -walks, and the only vertices in $V_D \cap V_{D'}$ they can reach by H -walks are precisely the vertices in $N_D^+(x)$. Thus, K' is independent by H -walks.

Now, suppose that K' is a kernel by H -walks of D' . We affirm that $K = K' \setminus S$ is a kernel of D . Let x be a vertex in $V_D \setminus K$. Since $T \cap K' = \emptyset$, it follows from Claim 1 that there exists a vertex $y \in N_D^+(x)$ such that $y \in K'$. Therefore, $y \in K$ and K is absorbent. Claim 1 together with the independence by H -walks of K' imply that K is an independent set of D .

Proof of Claim 1. Since there is an arc from red to green in H , for any arc $(x, y) \in A_D$, the walk $(x, x_{(x,y)}, y)$ is an H -walk in D' . Since the arcs from red to blue and from green to red are missing in H , the walk $(x, x_{(x,y)}, y)$ is a maximal H -walk in D' , and the walk $(x, v_{(x,y)}, v'_{(x,y)})$ is not an H -walk. \square

\square

Finally, we deal with digraphs H containing one of F_1, F_5, F_7 or F_8 .

Theorem 12. *Let H be a digraph containing F_i as an induced subdigraph for some $i \in \{1, 5, 7, 8\}$. The kernel by H -walks problem is NP -complete.*

Proof. Lemma 9 shows that the kernel by H -walks problem can be polynomially reduced from the k -coloring problem for graphs, for any $k \geq 3$. \square

We are now ready to prove our main result.

Proof of Theorem 1. Let H be a digraph which is not a panchromatic pattern. If H is not a looped digraph, Proposition 10 implies that the kernel by H -walks problem is NP -complete. Else, Theorem 5 implies that H contains an element of \mathcal{F} as an induced subdigraph. It follows from Theorems 11 and 12 that the kernel by H -walks problem is NP -complete. \square

6 Concluding remarks

We find interesting that, although for every digraph H containing F_7 or F_8 as an induced subdigraph, there exists an arc-colored digraph D such that D does not have a kernel by H -walks, no examples are known of such D . We think that once these examples are found, Theorem 12 could be improved by restricting the family of digraphs D where the kernel by H -walks problem remains NP -complete. This improvement can be achieved for digraphs H containing F_1 or F_5 as an induced subdigraph. Figure 3 shows an F_5 -arc-colored digraph without a kernel by F_5 -walks (on the left, Arpin and Linek [1]) and an F_1 -arc-colored digraph without a kernel by F_1 -walks (on the right). In both cases, the doubled arcs correspond to the top vertex of F_i , $i \in \{1, 5\}$, in Figure 1.

Corollary 13. *Let H be a digraph containing F_1 or F_5 as an induced subdigraph. The kernel by H -walks problem is NP -complete, even when restricted to bipartite digraphs with circumference $k \geq 6$.*

Sketch of proof. Notice that we can choose the digraph F_v used in the gadget for the vertex v of the reduction given in Section 4 to be one of the digraphs in Figure 3, let us call it F . Since both digraphs are bipartite, if we choose any even integer $k \geq 6$ in the reduction given in Section 4, then we can modify the construction of the vertex gadget to obtain a bipartite digraph

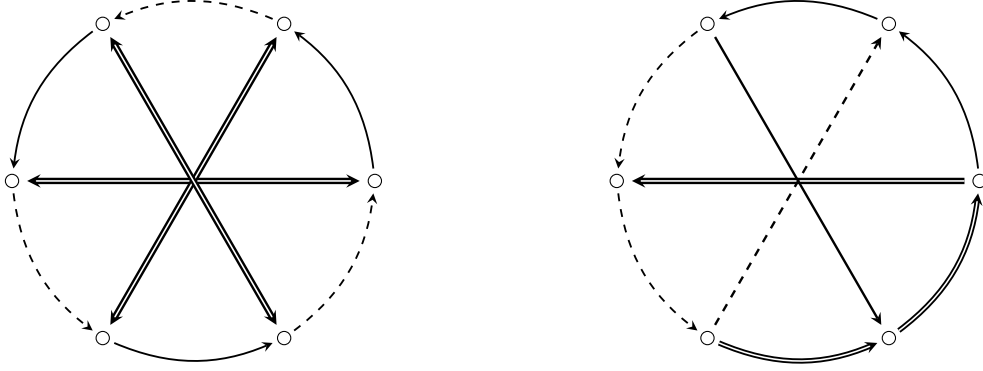


Figure 3: On the left, an F_5 -arc-colored digraph without a kernel by F_5 -walks. On the right, a bipartite tournament without a kernel by monochromatic paths.

(we do not need all the arcs from F to the k -cycle, so we can choose them accordingly to the bipartition of F and the k -cycle).

Notice that the gadget for every arc is a bipartite digraph, so the resulting digraph of the construction D' is a bipartite digraph. Clearly, the circumference of D' is k , so we have our desired result. \square

A result analogous for F_7 or F_8 would be obtained if an F_i -arc-colored bipartite digraph without a kernel by F_i -walks could be found, $i \in \{7, 8\}$. Moreover, the construction of the digraph D' in Section 4 is quite flexible, so results analogous to Corollary 13 with some other restrictions on the input could be obtained depending on the properties of those digraphs. We think that the following simple problem is interesting.

Problem 14. *Find an F_i -arc-colored (bipartite) digraph without a kernel by F_i -walks, $i \in \{7, 8\}$.*

It is also worth noticing that having more information on the structure of H might yield results restricting even more the structure of the input digraph. Consider the following result.

Theorem 15. *It is NP-complete to determine whether a digraph with 4-colored arcs has a kernel by monochromatic paths, even when restricted to planar digraphs with $\Delta^+, \Delta^- \leq 2$ and $\Delta \leq 3$.*

Sketch of proof. The reduction is from the kernel problem in planar graphs with the required degree constraints. Let D be an input digraph for the kernel problem. By Vizing's Theorem we can find a proper 4-coloring of the arcs of D to obtain a digraph with 4-colored set of arcs D' . Clearly, a subset S of V_D is a kernel of D if and only if it is a kernel by monochromatic paths of D' . \square

Now that the complexity of the kernel by H -walks problem has been completely classified, what else can be done in order to find sufficient conditions for the existence of kernels by H -walks? Of course it is useless to restrict the digraph H , and our previous remarks on the flexibility on the construction given in Section 4 intuitively say that restricting the structure of the input digraph D will not be very useful either. We think that it would be interesting to restrict the way the arcs of D are colored with V_H ; this idea has been previously explored in [9] and it may be worth considering again.

Probably, the most interesting problem we can think of right now is the following.

Problem 16. *Is there a dichotomy for the kernel by H -paths problem?*

It is not hard to verify that most of our reductions also work for the kernel by H -paths problem. But of course, we do not have a panchromatic pattern characterization for the path version of the problem. So that would be a good starting point.

References

- [1] P. Arpin and V. Linek, Reachability problems in edge-colored digraphs, Discrete Math. 307 (2007) 2276–2289.
- [2] J. Bang-Jensen and G. Gutin, Digraphs. Theory, Algorithms and Applications. Springer-Verlag, 2002.
- [3] J. Bang-Jensen, Y. Guo, G. Gutin and L. Volkmann, A classification of locally semicomplete digraphs, Discrete Math. 167/168 (1997) 101–114.
- [4] J.A. Bondy and U.S.R. Murty, Graph Theory, Springer-Verlag (2008).

- [5] V. Chvátal, On the computational complexity of finding a kernel, Technical Report Centre de recherches mathématiques, Université de Montréal, CRM-300, 1973.
- [6] A.S. Fraenkel, Planar kernel and Grundy with $d \leq 3, d^+ \leq 2, d^- \leq 2$ are NP-complete, Discrete Applied Mathematics 3 (1981) 257–262.
- [7] H. Galeana-Sánchez, On monochromatic paths and monochromatic cycles in edge colored tournaments, Discrete Math. 156 (1996) 103–112.
- [8] H. Galeana-Sánchez, Kernels in edge colored digraphs, Discrete Math. 184 (1998) 87–99.
- [9] H. Galeana-Sánchez, Kernels by monochromatic paths and the color-class digraph, Discussiones Mathematicae Graph Theory 31(2) (2011) 273–281.
- [10] H. Galeana-Sánchez, B. Llano and J.J. Montellano-Ballesteros, Kernels by monochromatic paths in m -colored unions of quasi-transitive digraphs, Discrete Applied Mathematics 158 (2010) 461–466.
- [11] H. Galeana-Sánchez and R. Strausz, On Panchromatic Patterns, Graphs and Combinatorics, Accepted.
- [12] G. Hanh, P. Ille and R. Woodrow, Absorbing sets in arc-colored tournaments, Discrete Math. 283 (2004) 93–99.
- [13] P. Hell, Graph partitions with prescribed patterns, European Journal of Combinatorics 35 (2014) 335–353.
- [14] P. Hell and C. Hernández-Cruz, On the complexity of the 3-kernel problem in some classes of digraphs, Discussiones Mathematicae Graph Theory 34(1) (2014) 167–185.
- [15] P. Hell and C. Hernández-Cruz, Minimal digraph obstructions for small matrices, arXiv (2016).
- [16] V. Linek and B. Sands, A note on paths in edge-colored tournaments, Ars Combin. 44 (1996) 225–228.

- [17] B. Sands, N. Sauer and R. Woodrow, On Monochromatic Paths in Edge-Coloured Digraphs, *Journal of Combinatorial Theory, Series B*, 33 (1982) 271–275.
- [18] M.G. Shen, On monochromatic paths in m -colored tournaments, *J. Combin. Theory Ser. B* 45(1) (1998) 108–111.